

Tunneling and Universality in the Integer Quantum Hall Effect

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We show analytically and numerically that omission of quantum interference from the Chalker–Coddington model of the integer quantum Hall effect gives a localization length exponent $\nu = 4/3$ as in ordinary two-dimensional percolation. Thus, contrary to semi-classical scaling arguments, tunneling alone does not lead to $\nu = 7/3$.

Scaling and universality in the integer quantum Hall effect (IQHE) related to the localization–delocalization transition of charge carriers when going from one plateau to the next is of considerable interest [1,2]. By measuring how the width of the peaks of the longitudinal conductance scales with magnetic field B and linear size W of the system, Koch et al. [3] found the scaling law $W \sim |B - B_c|^{-\nu}$ where $\nu \approx 2.3$ and B_c is the value of the magnetic field needed to have half filling.

This value excludes early theories [4,5] that predicted universal two-dimensional percolation exponent ($\nu = \nu_p = 4/3$). However, much before the measurements, Mil'nikov and Sokolov [6] argued on the basis of a semi-classical picture and scaling that the classical value should be shifted by unity, i.e. $\nu = 7/3$. Numerically 2.35 was found [7,8] in the Chalker–Coddington (CC) model [7], which includes in addition to the semi-classical tunneling effect also interference, and seems to capture the critical behavior in IQHE. It has been unclear whether tunneling is sufficient to explain, at least in a range of parameters, the change of the localization length exponent from the classical percolation value, $4/3$, to the observed one, ($\sim 7/3$) [9–11].

We present here a study of the effect of tunneling without interference in a CC type model. Our conclusions, based on analytic arguments and tested numerically, are as follows: 1) Tunneling alone does *not* change the localization length exponent from its classical percolation value, $4/3$. 2) The value actually observed is caused by the mechanism we left out (but emphasized by CC): Quantum interference.

In the classical picture the delocalization of the charge carriers is considered as a percolation transition [4,5]. Noninteracting charges are assumed to move in a disordered potential slowly varying on the scale of the magnetic length $l = \sqrt{\hbar/eB}$ (high field model). In this approximation, the electron wave function ψ is localized to equipotential curves, forming “ribbons” of width l . Changing B causes the localized wave function to move up or down in this landscape. Typically, the ribbons constituting the wave function are restricted to encircling mountains or valleys in the energy landscape. However, scattering between different ribbons is possible when they overlap at saddle points. There is a typical distance a between saddle points. In a window ΔB centered at B_c , a sample-spanning cluster of ribbons overlapping at saddle points exists, resulting in delocalization. The width of the window ΔB is determined by the magnetic length l and decreases with increasing sample size W . This classical description becomes semi-classical once the question of overlap of wave functions across saddle points takes tunneling into account. Overlap is then characterized by a tunneling probability [12]

$$P = \frac{1}{1 + e^{1/\chi}} \quad ; \quad \chi = \frac{ml^2 |V_{xx} V_{yy}|^{1/2}}{\hbar e |B - B_c|} . \quad (1)$$

Here V_{xx} and V_{yy} are the principal double spatial derivatives of the disordered potential at the saddle point, and m is the effective electron mass. The semi-classical argument leading to $\nu = \nu_p + 1 = 7/3$ is essentially based on a change of effective basic length scale of the percolation problem from a to χa . However, there is an underlying assumption that the tunneling does not change in any essential way the structure of the infinite cluster defined by the topology

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of the paths in the random energy landscape. As we will show later, this is the weak part of the argument, and the cause of its breakdown.

In order to proceed we study the semi-classical argument using the extended CC model [13]. We define first a fully classical (percolation) version [14]. Imagine a square lattice of linear size L where every plaquette is occupied by one of the tiles of Fig. 1. The result is a set of lines which are either closed or end at the boundary of the sample, see Fig. 2a. A control parameter p tells the probability of the tile 1a. This problem can be mapped exactly to bond percolation on a square lattice [14]. The mapping is accomplished by drawing a diagonal on each tile oriented in such a way that it does not intersect the two quarter circles already on the tile. The resulting pattern of diagonals is shown in Fig. 2a. This pattern consists of two lattices oriented 45° with respect to the original one defined by the tiles, and with lattice constant equal to the length of the diagonals of the tiles. The two lattices are displaced by a distance equal to the linear size of the tiles in the two principal orthogonal directions. Furthermore, whenever there is a bond present in one of the two lattices, the corresponding one is missing in the other lattice, and *vice versa*, making the two lattices duals of each other. This is shown in Figs. 2a, b and c. This makes the problem of random orienting the two tiles of Fig. 1 equivalent to placing a bond or not in one of the two intertwined lattices, say that of fig. 2b. It is also clear that, for $p = 1/2$ we deal with a percolation problem at the critical point, as with equal probability in choosing either tile corresponds to placing a bond with probability $1/2$, which is the percolation threshold on the square lattice.

The bonds of the tiling represent the classical paths the electrons may take in the disordered energy landscape. Each bond can only be traversed in one direction. The possible directions are chosen so that the network forms a grid of loops of alternating handedness according to the magnetic field and the surface topology. The centers of the tiles represent saddle points, and the lattice constant corresponds to a . The question of delocalization is equivalent to whether there is a path spanning the system.

We now introduce tunneling within the framework of a semi-classical approximation. The transmission probabilities at a given saddle point are controlled by a parameter γ characterizing the amount of tunneling [13]. By adjusting it to the usual control parameter of the CC model determining the scattering properties at the saddle point we have

$$\chi = \frac{1}{2|\ln \sinh \gamma|} . \quad (2)$$

When $\gamma \rightarrow 0$ or $\gamma \rightarrow \infty$, tunneling is negligible, while when $\gamma = \gamma_c = \ln(1 + \sqrt{2})$, the transmission probability is $1/2$.

Following Lee et al. [13], we assume $\gamma = \gamma_c e^{\mu - v}$. Here μ is the dimensionless chemical potential equal for all saddle points in the network, while v is chosen randomly on the interval $[-w/2, +w/2]$ representing the random potential. In the case when $|\mu - v| \ll 1$, we may relate $|\mu - v| \propto |B - B_c|$ of the high-field model. In extreme case when $w \rightarrow \infty$, there is no contact across the saddle points and the model is just the classical version discussed above with $1/2$ occupation probability for both tiles. Delocalization happens then as a classical percolation phenomenon and, accordingly, the correlation length exponent has to be $4/3$.

We now turn to the case of finite w when tunneling becomes important. In the CC model one should assign phases to the electrons moving in the channels and the change of the phases during tunneling should be taken into account. We neglect the phases here since the aim of this work is to test the effect of tunneling *alone*. What is the action of tunneling near a given saddle point, i.e. on a given tile in this Fig. 2a? It simply means that, with some probability, a tiling in the classical picture has to be changed to the other one for this particular electronic process. This corresponds in the bond percolation picture either to remove or to place a bond with some well defined probability that mimics Eqs. (1). Thus, instead of introducing a $\gamma = \gamma_c \exp(\mu - v)$ and relating it to the tunnel probability P by Eq. (2), we can divide the assigning construction of the conducting paths into two parts: one which places the “classical” paths, and then a second probability that models the tunneling. Given a control parameter $p \in [0, 1]$, we draw two random numbers r and ρ from a uniform distribution between zero and one for each bond. The bond is present if the following function is equal to one,

$$\pi(p, \Lambda; r, \rho) = \theta(p - r) + (\theta(r - p) - \theta(p - r))\theta(P - \rho) , \quad (3)$$

where, θ is the step function and P is defined in Eq. (1), with $1/\chi = \Lambda|p - r|$ and Λ being a scale factor. In words: Tunneling introduces a bond if there was no one classically and cancels the bond if it was present. This means that actual realizations of paths are followed and an average over such paths should finally be carried out.

The smaller Λ , the more important is tunneling. The idea behind Eq. (3) is simple. If $\Lambda \rightarrow \infty$, $P \rightarrow 0$, and Eq. (3) reduces to $\pi(p; r, \rho) = \theta(p - r)$, which is describing a simple bond percolation problem with p as control parameter. We may therefore interpret r as characterizing the level of the corresponding saddle point, and p the magnetic field. The second term in Eq. (3) will switch the value of π given by the first “classical” term to the opposite value with a

probability which falls off exponentially the further p is away from the value r . In particular, when $p = r$ for bond i , there is a 50% chance that the value of π will be reversed compared to its classical value. Furthermore, the further r and p are apart, the more difficult to switch the value of π . This completes the definition of our tunnel-only version of the Chalker-Coddington model.

The probability that a bond is present is then

$$\bar{\pi}(p) = \langle \pi(p; r, \rho) \rangle = \int_0^1 dr \int_0^1 d\rho \pi(p; r, \rho) = p + \frac{1}{\Lambda} \log \left(\frac{1 + e^{-\Lambda p}}{1 + e^{-\Lambda(1-p)}} \right), \quad (4)$$

Since assigning a bond or not is a purely local decision, which is so far based on two random numbers r and ρ , we may simplify and draw a single random number q between zero and one and comparing this to $\bar{\pi}$:

$$\pi(p; q) = \theta(\bar{\pi} - q). \quad (5)$$

This formulation is equivalent to that of Eq. (3). It is simpler to analyze, but harder to interpret. The effect of tunneling has been to produce a non-linear relationship between the control parameter p and the probability for a bond to be present. However, we have that $\bar{\pi} = 1/2$ when $p = 1/2$, as expected: The system has the same critical point with or without tunneling: $p_c = 1/2$.

We are now ready to analyze what happens close to the critical value of p , and in particular determining the localization length exponent ν in this model. In order to do this, we first note that using $\bar{\pi}$ as control parameter and not p , makes the problem a standard percolation problem. Thus, the correlation length ξ diverges as

$$\xi \sim |\bar{\pi}_c - \bar{\pi}|^{-\nu_p}, \quad (6)$$

where $\bar{\pi}_c = 1/2$, and $\nu_p = 4/3$. We now expand $\bar{\pi}$ in powers of $\delta p = p_c - p$ to get

$$\bar{\pi}(1/2 + \delta p) = 1/2 + \tanh \left(\frac{\Lambda}{4} \right) \delta p + \mathcal{O}(\delta p^3). \quad (7)$$

Plugging this expansion into Eq. (7) gives

$$\xi \sim |p_c - p|^{-\nu_p}, \quad (8)$$

but with a different prefactor. The exponent in this equation is by definition ν and we have shown that

$$\nu = \nu_p = \frac{4}{3} \quad (9)$$

in the CC model when only tunneling is taken into account. Based on this result, we conclude that the semi-classical argument giving $\nu = \nu_p + 1$ fails.

It is interesting to note that in a one-dimensional version of this model, first studied in [11], there *is* a shift of ν in comparison to $\nu_p = 1$ by precisely one [15]. Thus, in this case, the semiclassical argument *does* work. The difference between the one and two-dimensional cases is that in the latter the number of bonds almost belonging to the infinite cluster is so large — their density is finite — that the system simply is driven away from criticality.

We have carried out numerical simulations of the semi-classical CC model as defined above in order to see the way asymptotics sets in. We generated different systems sizes up to 1100×1100 with at least 20 000 samples using Monte Carlo renormalization group [16] to determine the exponents. We have based our simulations on Eq. (5) directly, and on Eq. (3) directly. In the latter case, we used *geometrical* sample averaging in order to test whether the *typical* percolation probability behaves differently from the average one. We found no difference in the critical behavior. The measured effective exponent was $\nu = 1.53$ for $W = 100$ and $\nu = 1.45$ for $W = 1100$. Thus there is a slow convergence towards the predicted universal exponent $4/3$ but even small size samples have an effective exponent much smaller than $7/3$, confirming our analytical conclusion.

The localization length exponent observed in the Chalker-Coddington model and in other models of the integer quantum Hall effect in addition to experiments, is larger than ν_p by about unity. What we have shown here is that the tunneling mechanism alone is not able to account for the change of the exponent in any range of the parameters — therefore interference must be responsible for it.

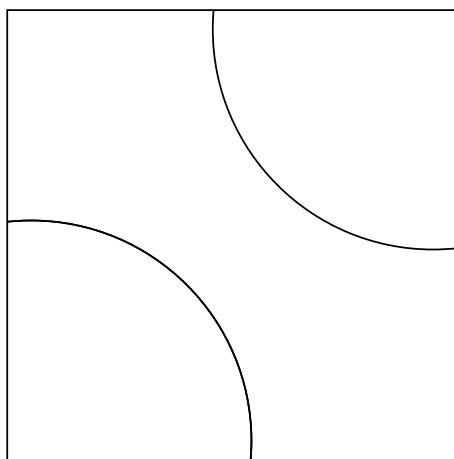
As the control parameter ΔB increases the interference becomes less and less pronounced and the entangled paths due to tunneling percolation dominate. This should be reflected in a crossover from the quantum exponent $\nu \approx 2.3$ to the classical value $4/3$. We suggest the experimental check of this prediction.

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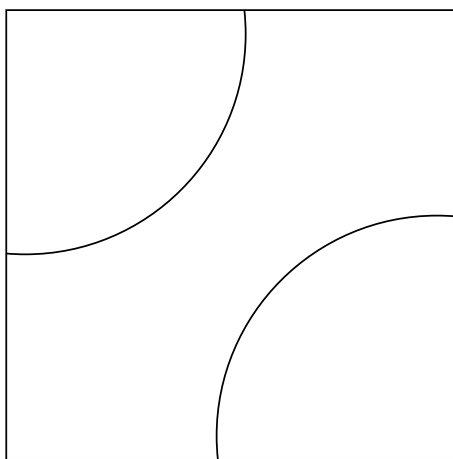
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FIG. 1. The two fundamental tiles constituting the building blocks of the CC model in the classical limit.

FIG. 2. (a) A set of dragon curves with corresponding diagonals inscribed for $p = 1/2$. In (b) and (c) we have decomposed the set of diagonals into a sublattice (b) and the corresponding dual lattice (c).



a



b

Figure 1

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Tunneling and Universality in the Integer Quantum Hall Effect

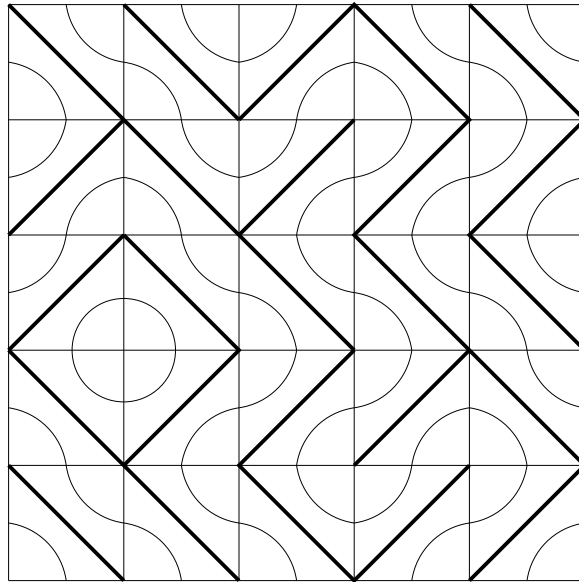


Figure 2a

A. Hansen and J. Kertesz

Tunneling and Universality in the Integer Quantum Hall Effect

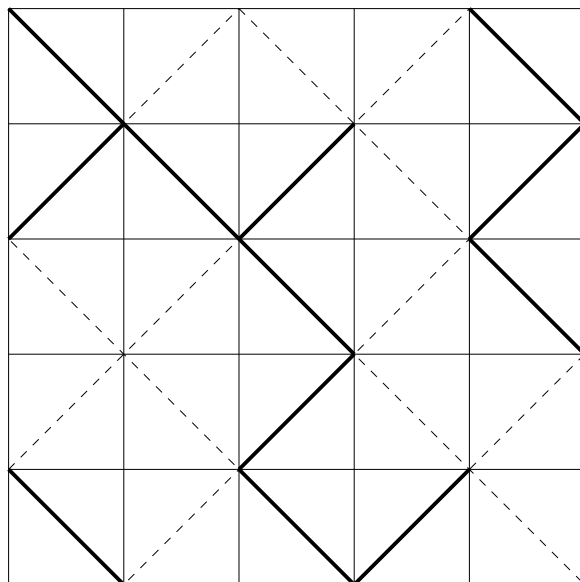


Figure 2b

A. Hansen and J. Kertesz

Tunneling and Universality in the Integer Quantum Hall Effect

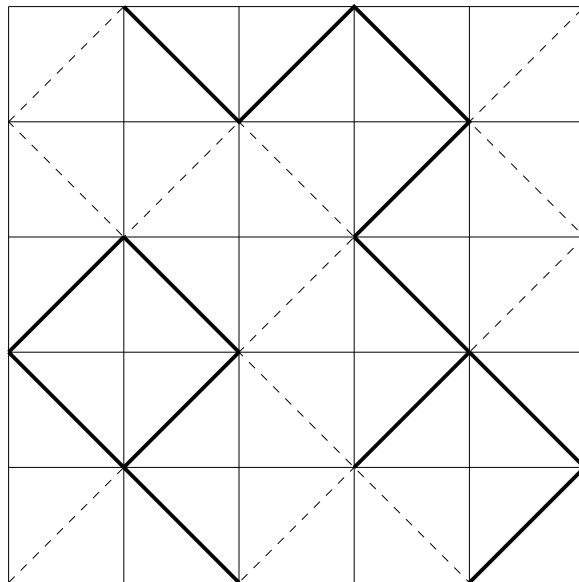


Figure 2c

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Tunneling and Universality in the Integer Quantum Hall Effect